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Statistics of two-dimensional random walks, the "cyclic sieving phenomenon" and the Hofstadter model

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Abstract

We focus on the algebraic area probability distribution of planar random walks on a square lattice with m_1, m_2, l_1 and l_2 steps right, left, up and down. We aim, in particular, at the algebraic area generating function $Z_{m_1,m_2,l_1,l_2}(\mathrm{Q})$ evaluated at $\mathrm{Q} = \mathrm{e}^{\frac{2i\pi}{q}}$, a root of unity, when both $m_1 - m_2$ and $l_1 - l_2$ are multiples of q. In the simple case of staircase walks, a geometrical interpretation of $Z_{m,0,l,0}(\mathrm{e}^{\frac{2i\pi}{q}})$ in terms of the cyclic sieving phenomenon is illustrated. Then, an expression for $Z_{m_1,m_2,l_1,l_2}(-1)$, which is relevant to the Stembridge's case, is proposed. Finally, the related problem of evaluating the nth moments of the Hofstadter Hamiltonian in the commensurate case is addressed.

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1 Introduction

The cyclic sieving phenomenon [1] is quite ubiquitous in combinatorics, whereby some finite sets have both a cyclic symmetry and a generating function which, when evaluated at roots of unity, happens to count some symmetry classes of the sets. A paramount example is the collection of the l-subsets of a $(m+l)$-set with the Q-binomial generating function

$$\binom{m+l}{l}_{\mathrm{Q}} \equiv \frac{[m+l]_{\mathrm{Q}}!}{[m]_{\mathrm{Q}}![l]_{\mathrm{Q}}!}\,, \tag{1}$$

The Q-factorial is defined as

$$[l]_{\mathrm{Q}}! = \prod_{i=1}^{l} \frac{1-\mathrm{Q}^i}{1-\mathrm{Q}} = 1(1+\mathrm{Q})(1+\mathrm{Q}+\mathrm{Q}^2)\cdots(1+\mathrm{Q}+\ldots+\mathrm{Q}^{l-1})\,. \tag{2}$$

Denoting by c the cycling generator (one-step cyclic permutation of the $(m+l)$ set), one finds that $\binom{m+l}{l}_{\mathrm{Q}}$ evaluated at $\mathrm{Q} = e^{2\mathrm{i}\pi p/(m+l)}$ counts the number of l-subsets fixed by (i.e., invariant with respect to) c^p for $p = 1, 2, \ldots, m+l$. In the particular case $\mathrm{Q} = -1$ one refers to the Stembridge's phenomenon [2]. The fact that integers show up here stems from a well-known identity: for any integers p, q mutually prime[4] and m a multiple of q, one has

$$\binom{m+l}{m}_{\mathrm{e}^{\frac{2\mathrm{i}\pi p}{q}}} = \binom{\left[\frac{m+l}{q}\right]}{\frac{m}{q}}\,. \tag{3}$$

A planar random walk on a square lattice is defined as an ordered sequence of steps to the right x, left x^{-1}, up y, and down y^{-1}, with numbers of steps m_1, m_2, l_1 and l_2, respectively. We will refer to such a walk as an (m_1, m_2, l_1, l_2) walk. The walk is closed if $m_1 = m_2 = m$ and $l_1 = l_2 = n/2 - m$, where n is the total number of steps (which is necessarily even), and open otherwise. We extend the standard definition of the algebraic area enclosed by a closed walk onto an open walk, as follows: close an open walk by connecting its end point with its start point, adding on to the end of the walk the minimum necessary number of steps, first vertical, then horizontal. E.g., if $m_1 \geq m_2$ and $l_1 \geq l_2$, then we close the walk by adding $l_1 - l_2$ steps down followed by $m_1 - m_2$ steps to the left. The area of the open walk is defined as the area of the closed walk thus obtained; essentially, it is the algebraic area *under* the open walk.

The generating function $Z_{m_1,m_2,l_1,l_2}(\mathrm{Q})$ of the algebraic area probability distribution of (m_1, m_2, l_1, l_2) walks originating from a given point on the lattice is defined in terms of the number $C_{m_1,m_2,l_1,l_2}(A)$ of such walks with algebraic area A:

$$Z_{m_1,m_2,l_1,l_2}(\mathrm{Q}) = \sum_{A=-\infty}^{\infty} C_{m_1,m_2,l_1,l_2}(A)\,\mathrm{Q}^A\,. \tag{4}$$

[4]In the sequel, Q denotes the argument of the Q-binomial, whereas the integer q denotes the root of unity $\mathrm{Q} = \exp(2\mathrm{i}\pi/q)$ at which the generating functions are evaluated.

$Z_{m_1,m_2,l_1,l_2}(\mathrm{Q})$ can be viewed as a generalization of the Q-binomial coefficient: If x and y (identified with steps to the right and up, respectively) satisfy the commutation relation $xy = \mathrm{Q}yx$, then

$$(x + y + x^{-1} + y^{-1})^n = \sum_{\substack{m_1,m_2,l_1,l_2 \\ m_1+m_2+l_1+l_2=n}} Z_{m_1,m_2,l_1,l_2}(\mathrm{Q}) y^{-l_1} y^{l_2} x^{m_1} x^{-m_2} \; . \tag{5}$$

It can be seen easily that $Z_{m_1,m_2,l_1,l_2}(\mathrm{Q})$ satisfies the recurrence relation

$$\begin{aligned} Z_{m_1,m_2,l_1,l_2}(\mathrm{Q}) = {} & Z_{m_1,m_2,l_1-1,l_2}(\mathrm{Q}) + Z_{m_1,m_2,l_1,l_2-1}(\mathrm{Q}) \\ & + \mathrm{Q}^{l_2-l_1} Z_{m_1-1,m_2,l_1,l_2}(\mathrm{Q}) + \mathrm{Q}^{l_1-l_2} Z_{m_1,m_2-1,l_1,l_2}(\mathrm{Q}) \; , \end{aligned} \tag{6}$$

with the initial condition $Z_{0,0,0,0}(\mathrm{Q}) = 1$. It is therefore a polynomial in Q and Q^{-1}, the solution of Eq. (6), but whose explicit expression remains unknown.

Trivially, when evaluated at $\mathrm{Q} = 1$,

$$Z_{m_1,m_2,l_1,l_2}(1) = \binom{m_1 + m_2 + l_1 + l_2}{m_1, m_2, l_1, l_2} \tag{7}$$

is the usual multinomial coefficient that counts the number of (m_1, m_2, l_1, l_2) walks.

Less trivially, for "biased walks", i.e., ones that can go right, up and down, but not left — that is to say with $m_2 = 0$ — the generating function $Z_{m_1,0,l_1,l_2}(\mathrm{Q})$ has been found [3] to be

$$\begin{aligned} Z_{m_1,0,l_1,l_2}(\mathrm{Q}) = & \sum_{k=0}^{\min(l_1,l_2)} \left[\binom{m_1 + l_1 + l_2}{k} - \binom{m_1 + l_1 + l_2}{k-1} \right] \\ & \binom{m_1 + l_1 - k}{m_1}_{\mathrm{Q}^{-1}} \binom{m_1 + l_2 - k}{m_1}_{\mathrm{Q}} \; . \end{aligned} \tag{8}$$

Again, when evaluated at $\mathrm{Q} = 1$, the multinomial counting for the number of $(m_1, 0, l_1, l_2)$ biased walks is recovered:

$$Z_{m_1,0,l_1,l_2}(1) = \binom{m_1 + l_1 + l_2}{m_1, l_1, l_2} \; . \tag{9}$$

When also $l_2 = 0$, $Z_{m_1,0,l_1,0}(\mathrm{Q})$ yields, as it should, the Q-binomial generating function

$$Z_{m,0,l,0}(\mathrm{Q}) = \binom{m + l}{m}_{\mathrm{Q}} \tag{10}$$

for the probability distribution of the algebraic area under staircase walks, ones that can go only m steps right and l steps up. One notes that this Q-binomial has been already introduced in Eq. (1) and evaluated at Q a root of unity in Eq. (3) in the context of the cyclic sieving phenomenon for the l-subsets of the $(m + l)$-set.

These considerations evoke a natural question: Can $Z_{m_1,0,l_1,l_2}(\mathrm{Q})$ in Eq. (8) with Q a root of unity be an integer, and if so, what does this integer count? More generally, for

(m_1, m_2, l_1, l_2) walks, despite $Z_{m_1,m_2,l_1,l_2}(\mathtt{Q})$ being generally unknown, can it be evaluated at $\mathtt{Q}$ a root of unity; if so, can it yield, at least in certain cases, an integer; if yes, what does this integer count?

Apart from these cyclic sieving combinatorics/counting considerations, $Z_{m_1,m_2,l_1,l_2}(\mathtt{Q})$ happens to be of interest not only for random walks but also for the quantum Hofstadter problem [4], thanks to the formal mapping [5] between the algebraic area generating function for closed random walks of length n

$$Z_n(\mathtt{Q}) = \sum_{m=0}^{n/2} Z_{m,m,\frac{n}{2}-m,\frac{n}{2}-m}(\mathtt{Q}) \tag{11}$$

and the n-moments of the Hofstadter Hamiltonian H_γ

$$Z_n(e^{i\gamma}) = \mathrm{Tr}\, H_\gamma^n \ , \tag{12}$$

where $\gamma = 2\pi\phi/\phi_0$ is the flux per plaquette in units of the flux quantum and $\mathtt{Q}$ has been taken to be $e^{i\gamma}$. Of particular interest is the commensurate case $\gamma = 2\pi p/q$, with p and q relative primes. Therefore, another motivation in evaluating $Z_{m_1,m_2,l_1,l_2}(\mathtt{Q})$ when $\mathtt{Q}$ is a root of unity stems from the Hofstadter quantum spectrum itself for a commensurate flux.

In this regard, for "closed" biased walks of length n — "closed" here being defined by $l_1 = l_2 = (n - m_1)/2$ and the choice of particular boundary conditions on the horizontal axis — an explicit quantum mapping has indeed been found [6]:

$$\sum_{m=0}^{[\frac{n}{q}]} Z_{qm,0,\frac{n-qm}{2},\frac{n-qm}{2}}(e^{i\gamma}) = \mathrm{Tr}\, \tilde{H}_\gamma^n, \quad \gamma = 2\pi/q \ , \tag{13}$$

where $\tilde{H}_\gamma$ now stands for a truncated Hofstadter Hamiltonian with the horizontal hopping to the left absent. In the sum above, each $Z_{qm,0,(n-qm)/2,(n-qm)/2}$ has to be understood with n and qm being of the same parity (otherwise it has to be taken equal to 0). Note also that the number of right steps qm is a multiple of q, so that one can view the "closed" biased walks as winding on a cylinder with circumference q.

2 The sieving phenomenon and staircase random walks

As a warm-up, let us rephrase the staircase random walk algebraic area generating function $Z_{m,0,l,0}(\mathtt{Q})$ evaluated at $\mathtt{Q}$ a root of unity in the context of the cyclic sieving phenomenon.

One first notes that an $(m, 0, l, 0)$ staircase walk consisting of m steps to the right and l steps up uniquely corresponds to an l-subset of the set $\{1, \ldots, m + l\}$. For example, $xxyxxy$, which is one of the possible $(4, 0, 2, 0)$ walks, corresponds to the subset $\{3, 6\}$ of the set $\{1, \ldots, 6\}$. A cyclic permutation c of the above-mentioned set is $1 \to 2 \to 3 \to \ldots \to m + l \to 1$. The cyclic sieving phenomenon is the statement that $Z_{m,0,l,0}(\mathrm{e}^{\frac{2\mathrm{i}\pi p}{m+l}})$ equals the number of the subsets fixed by c^p for $p = 1, 2, \ldots, m + l$.

Next, one notes that when both m and l are multiples of q, Eq. (3) implies

$$Z_{m,0,l,0}(\mathrm{e}^{\frac{2\mathrm{i}\pi}{q}}) = Z_{\frac{m}{q},0,\frac{l}{q},0}(1) \;, \tag{14}$$

and the quantity on the RHS counts the number of staircase walks with m/q steps to the right and l/q steps up.

This counting allows for a simple interpretation in terms of the staircase walk $(m,0,l,0)$. The most general l-element subset of the set $\{1,\ldots,m+l\}$ fixed by c^p has the form $\{i_1, i_2, \ldots, i_{l/q}, i_1+p, i_2+p, \ldots, i_{l/q}+p, \ldots, i_1+(q-1)p, i_2+(q-1)p, \ldots, i_{l/q}+(q-1)p\}$, where $q=(m+l)/p$ (i.e., q copies of a subset $\{i_1, i_2, \ldots, i_{l/q}\}$, with $i_{l/q} \le p$, shifted by p with respect to each other). A subset $\{i_1, i_2, \ldots, i_{l/q}\}$ corresponds, as formulated above, to a $(\frac{m}{q}, 0, \frac{l}{q}, 0)$ walk. Thus, any $(m,0,l,0)$ walk corresponding to a subset fixed by c^p is a repetition of q identical "building blocks", shifted with respect to each other by p steps. Equation (14) expresses the fact that the total number of such walks is equal to the number of possible building blocks.

Consider an example: $m=9$, $l=3$. The values of $Z_{9,0,3,0}(\mathrm{e}^{\frac{2\mathrm{i}\pi p}{12}})$ with p ranging from 1 to 12 are: $\{0, 0, 0, 4, 0, 0, 0, 4, 0, 0, 0, 220\}$. The first 4 in this sequence, corresponding to $p=4$, counts the number of ways that a $(9,0,3,0)$ walk can be constructed from 3 building blocks, each block being a $(3,0,1,0)$ walk: there are 4 such blocks possible. Each of those walks is fixed by c^4. The second 4, corresponding to $p=8$, reflects the fact that any subset fixed by c^4 is also, trivially, fixed by c^8. Finally, every walk is fixed by $c^{12}=I$, so the 220 is the total number of walks $\binom{12}{3}$. The zero values of $Z_{m,0,l,0}(\mathrm{e}^{\frac{2\mathrm{i}\pi p}{m+l}})$ reflect the fact that there are no walks fixed by c, c^2, c^3, c^5, etc.

Note also that the non-zero values can be directly retrieved from $Z_{m,0,l,0}(\mathrm{e}^{\frac{2\mathrm{i}\pi p}{q}})$ with $p=1,2,\ldots,q$ where q is the GCD of m and l. In the example above $Z_{9,0,3,0}(\mathrm{e}^{\frac{2\mathrm{i}\pi p}{3}})$ with $p=1,2,3$ directly yields $\{4, 4, 220\}$.

3 The Stembridge phenomenon for biased and unbiased random walks

Consider now in Eq. (8) the generating function $Z_{m_1,0,l_1,l_2}(\mathrm{Q})$ for the algebraic area distribution of biased walks and evaluate it with Q a root of unity. Looking either at the staircase walks counting in Eq. (3) with m a multiple of q or at the q-periodic sum in Eq. (13), let m_1 be a multiple of q. It follows immediately that $Z_{m_1,0,l_1,l_2}(\mathrm{e}^{\frac{2\mathrm{i}\pi}{q}})$ is an integer. Furthermore, as for staircase walks[5], all $Z_{m_1,0,l_1,l_2}(\mathrm{e}^{\frac{2\mathrm{i}\pi p}{q}})$'s for $p=1,2,\ldots,q$ are also non-vanishing integers.

One can go a step further by having not only m_1 but also $|l_1-l_2|$ be a multiple of q,

[5]The analogy stops here: $Z_{m_1,0,l_1,l_2}(\mathrm{e}^{\frac{2\mathrm{i}\pi p}{m_1+l_1+l_2}})$ with $p=1,2,\ldots,m_1+l_1+l_2$ would yield non-integer values for certain values of p.

as suggested by Eq. (14). Some algebra allows then to rewrite $Z_{m_1,0,l_1,l_2}(\mathrm{e}^{\frac{2\mathrm{i}\pi}{q}})$ as

$$Z_{m_1,0,l_1,l_2}(\mathrm{e}^{\frac{2\mathrm{i}\pi}{q}}) = \sum_{k=\min(l_1,l_2)}^{0,-q} \binom{m_1+l_1+l_2}{k} \frac{m_1(m_1+l_1+l_2-2k)}{(m_1+l_1-k)(m_1+l_2-k)} \binom{\frac{m_1+l_1-k}{q}}{\frac{m_1}{q}} \binom{\frac{m_1+l_2-k}{q}}{\frac{m_1}{q}} . \tag{15}$$

In Eq. (15), $\sum_{k=\min(l_1,l_2)}^{0,-q}$ means that the sum over k is from $\min(l_1,l_2)$ down to 0 by steps of minus q — from which one deduces that the entries of the last two binomials are indeed integers. It is also understood that when $m_1 = 0$, the ratio $\frac{m_1(m_1+l_1+l_2-2k)}{(m_1+l_1-k)(m_1+l_2-k)}$ is non-vanishing and actually equal to 1 only when $k = \min(l_1,l_2)$. This yields, as it should, $Z_{0,0,l_1,l_2}(\mathrm{e}^{\frac{2\mathrm{i}\pi}{q}}) = \binom{l_1+l_2}{\min(l_1,l_2)} = \binom{l_1+l_2}{l_1}$, the number of (l_1,l_2) walks on the vertical axis which all have, trivially, a vanishing algebraic area so that $Z_{0,0,l_1,l_2}(\mathrm{e}^{\frac{2\mathrm{i}\pi}{q}}) = Z_{0,0,l_1,l_2}(1)$.

In (15) for each[6] k the first binomial counts the number of ways to pick k elements out of the $(m_1+l_1+l_2)$-element set, whereas the last two binomials are the numbers of $(\frac{m_1}{q}, \frac{l_1-k}{q})$ and $(\frac{m_1}{q}, \frac{l_2-k}{q})$ staircase walks, respectively. Finally, the ratio in (15) encodes the fact that the l_1-k steps up and l_2-k steps down "annilihate" each other on the vertical axis so that these staircase walks cannot be considered as independent.

Now, let us focus on the Stembridge's case $\mathtt{Q} = -1$, that is to say $q = 2$: one can then simplify (15) to get the quite convincing expression

$$Z_{m_1,0,l_1,l_2}(-1) = \binom{l_1+l_2}{l_1} \binom{\frac{m_1+l_1+l_2}{2}}{\frac{l_1+l_2}{2}} . \tag{16}$$

Furthermore, one can go a step further and relax the biased constraint $m_2 = 0$, that is to say consider now more general (m_1,m_2,l_1,l_2) walks, with both $|m_1-m_2|$ and $|l_1-l_2|$ multiples of q. Again, in the Stembridge's case $q = 2$, one obtains

$$Z_{m_1,m_2,l_1,l_2}(-1) = \binom{m_1+m_2}{m_1} \binom{l_1+l_2}{l_1} \binom{\frac{m_1+m_2+l_1+l_2}{2}}{\frac{l_1+l_2}{2}} , \tag{17}$$

which does reduce to Eq. (16) for $m_2 = 0$ biased walks.

Eq. (17), and consequently Eq. (16), yields integers, since $|m_1-m_2|$ and $|l_1-l_2|$ being even implies that m_1+m_2 and l_1+l_2 are even as well. A Stembridge combinatorial

[6] Note that for each such k,

$$\frac{m_1(m_1+l_1+l_2-2k)}{(m_1+l_1-k)(m_1+l_2-k)} \binom{\frac{m_1+l_1-k}{q}}{\frac{m_1}{q}} \binom{\frac{m_1+l_2-k}{q}}{\frac{m_1}{q}}$$

is by itself an integer, which is nothing but saying that, trivially,

$$\frac{a(a+b+c)}{(a+b)(a+c)} \binom{a+b}{a} \binom{a+c}{a}$$

is an integer for any integers a, b, c.

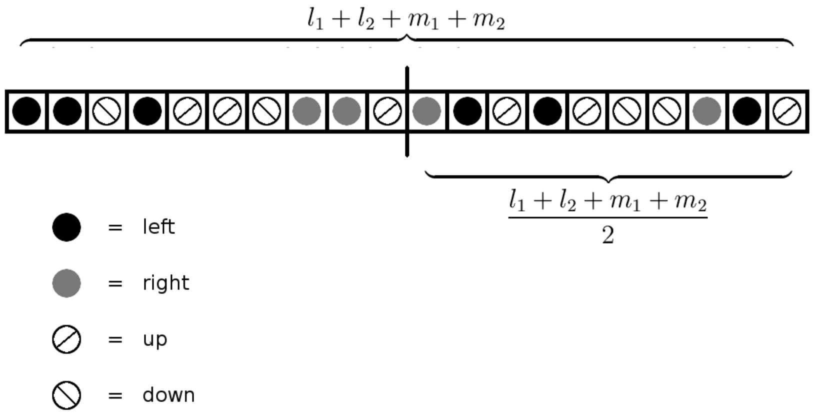

Figure 1: A Stembridge interpretation.

interpretation of these integers is depicted in Fig. 1. Divide the $(m_1+m_2+l_1+l_2)$ cells into two equal subsets ($q=2$, i.e., two building blocks). Then consider four types of objects (corresponding to four directions — up, down, left, right) and define a subset as "fixed in the weak sense" by $c^{(m_1+m_2+l_1+l_2)/2}$ — meaning that when acting by c, "up" is considered to be the same as "down" and "left" the same as "right". The formula is then interpreted as follows: the rightmost factor counts the number of ways that half of the (up + down) objects, $(l_1+l_2)/2$, can be distributed between half the cells, $(m_1+m_2+l_1+l_2)/2$. Then two other factors distinguish between "up" and "down" (l_1 out of l_1+l_2) and between "left" and "right" (m_1 out of m_1+m_2).

The simple expression of $Z_{m_1,m_2,l_1,l_2}(\mathrm{e}^{\frac{2\mathrm{i}\pi}{2}})$ in Eq. (17) asks for a generalization to $Z_{m_1,m_2,l_1,l_2}(\mathrm{e}^{\frac{2\mathrm{i}\pi}{q}})$ for (m_1,m_2,l_1,l_2) walks with both $|m_1-m_2|$ and $|l_1-l_2|$ multiples of q. Such an expression for $Z_{m_1,m_2,l_1,l_2}(\mathrm{e}^{\frac{2\mathrm{i}\pi}{q}})$ would in turn allow one to retrieve the n-moments of the Hofstadter Hamiltonian at commensurate flux by summing $Z_{m,m,l=n/2-m,l=n/2-m}(\mathrm{Q})$ over m, like in Eq. (11) — with in that case both $|m_1-m_2|$ and $|l_1-l_2|$ vanishing, in principle an even more simple situation — and then using Eq. (12). Following this line of reasoning for $q=2$, i.e., the $\mathrm{Q}=-1$ Stembridge case, one eventually gets from Eq. (17) the nth moment of H_π

$$\mathrm{Tr}\, H_\pi^n = \sum_{m=0}^{n/2} Z_{m,m,l=n/2-m,l=n/2-m}(-1) = \sum_{m=0}^{n/2} \binom{2m}{m}\binom{2(\frac{n}{2}-m)}{\frac{n}{2}-m}\binom{\frac{n}{2}}{m}\,. \tag{18}$$

This last result is a sum rule for the Hofstadter spectrum at commensurate flux $\gamma=\pi$. Below we compute $\mathrm{Tr}\, H_{2\pi/q}^n$ for particular values of $q=2,3,4,6$ directly from the quantum

mechanics formulation (i.e., without relying on an evaluation of $Z_{m,m,n/2-m,n/2-m}(\mathrm{e}^{\frac{2i\pi}{q}})$).

4 The nth moments of the Hofstadter Hamiltonian

Up to now we have used the mapping of the quantum Hofstadter problem on classical random walks to arrive at the Hofstadter sum rule (18). This sum rule can be directly retrieved from the quantum Hofstadter Hamiltonian itself, as can be other sum rules for simple values of $q = 3, 4, 6$.

For concreteness, we wish to compute the nth moment

$$\mathrm{Tr}\, H_\gamma^n \tag{19}$$

where H_γ is the Hofstadter Hamiltonian

$$H_\gamma = x + x^{-1} + y + y^{-1} \ , \quad xy = e^{i\gamma} yx \ ; \tag{20}$$

x and y stand now for the quantum lattice hopping operators on the horizontal and vertical directions respectively: they do non commute, due to the presence of the perpendicular magnetic field with flux γ per unit cell. The definition of trace in (19) is such that

$$\mathrm{Tr}\, (x^m y^n) = \delta_{m,0}\, \delta_{n,0} \ . \tag{21}$$

We define the generating function

$$\mathcal{Z}(\mathrm{e}^{\mathrm{i}\gamma}, t) = \mathrm{Tr} \sum_{n=0}^{\infty} H_\gamma^n\, t^n = \mathrm{Tr}\, \frac{1}{1 - tH_\gamma} = \mathrm{Tr}\, \frac{1}{1 - t^2 H_\gamma^2} \tag{22}$$

where the last rewriting follows from the vanishing of $\mathrm{Tr}\, H_\gamma^n$ for odd n.

From now on we focus on the commensurate case $\gamma = 2\pi/q$, for which the operators x^q and y^q do commute with x and y and are Casimirs, and so is

$$C_q = x^q + x^{-q} + y^q + y^{-q} \ . \tag{23}$$

C_q is, essentially, the Hofstadter Hamiltonian with $q = 1$, i.e., $\gamma = 2\pi$, that is to say zero flux. One easily obtains

$$\mathrm{Tr}\, C_q^{2n} = \binom{2n}{n}^2 \tag{24}$$

and

$$\begin{aligned} \mathrm{Tr}\, H_{2\pi/q}^2 &= 4 \ , \\ \mathrm{Tr}\, H_{2\pi/q}^4 &= 28 + 8\cos(2\pi/q) \ , \\ \mathrm{Tr}\, C_q^{2n+1} &= \mathrm{Tr}\, H_{2\pi/q}^{2n+1} = 0 \ , \end{aligned} \tag{25}$$

so $\mathrm{Tr}\, H_{2\pi/q}^n$ will be nonzero only for even values of n, as expected.

The operators x, y can be represented as $q \times q$ matrices (with x^q and y^q proportional to the identity matrix) and therefore so can $H_{2\pi/q}$. As a result, it satisfies a characteristic

equation of degree q, and so $H_{2\pi/q}^q$ can be expressed in terms of lower powers of $H_{2\pi/q}$ and the Casimir C_q. Defining the "parity" of the monomial $x^m y^n$ as $(-1)^{m+n}$, $H_{2\pi/q}^n$ has parity $(-1)^n$ and only like-parity powers of $H_{2\pi/q}$ will appear in the expression of $H_{2\pi/q}^q$. Overall we have

$$H_{2\pi/q}^q = C_q + 2q H_{2\pi/q}^{q-2} + c_{q-4} H_{2\pi/q}^{q-4} + \cdots + c_{q (\mathrm{mod}\ 2)} H_{2\pi/q}^{q (\mathrm{mod}\ 2)} \ , \tag{26}$$

where c_k are q-dependent numerical coefficients. Looking back at random walks, this is a rewriting of Eq. (5) but with the different terms regrouped into powers of $H_{2\pi/q}$ itself, which is possible for a rational flux $\gamma = 2\pi/q$. The term C_q arises out of the terms x^q, x^{-q}, y^q, y^{-q} in the expansion of $H_{2\pi/q}^q$. The fact that the coefficient c_{q-2} of the term $H_{2\pi/q}^{q-2}$ is $2q$ can be proven using combinatorics and the identity $1 + e^{2\mathrm{i}\pi/q} + \cdots + (e^{2\mathrm{i}\pi/q})^{q-1} = 0$.

Performing the above rewriting explicitly for the first few values of q, we obtain

$$\begin{aligned}
q=2: &\quad H_{\pi}^2 &&= 4 + C_2 \\
q=3: &\quad H_{2\pi/3}^3 &&= 6H_{2\pi/3} + C_3 \\
q=4: &\quad H_{\pi/2}^4 &&= 8H_{\pi/2}^2 + C_4 - 4 \\
q=5: &\quad H_{2\pi/5}^5 &&= 10H_{2\pi/5}^3 - (5 - 2\cos\frac{2\pi}{5}) H_{2\pi/5} + C_5 \\
q=6: &\quad H_{\pi/3}^6 &&= 12H_{\pi/3}^4 - 24H_{\pi/3}^2 + 4 + C_6
\end{aligned} \tag{27}$$

The above relations will allow for the exact evaluation of $\mathrm{Tr}\ H_{2\pi/q}^n$. The zero flux case $q = 1$, i.e., $\gamma = 2\pi$ (corresponding to the unweighted random walk, $\mathtt{Q} = 1$) is explicitly solved as

$$\mathrm{Tr}\ H_{2\pi}^{2n} = \binom{2n}{n}^2, \quad \mathcal{Z}(1,t) = \frac{2}{\pi} K(4t) \tag{28}$$

with $K(k)$ the complete elliptic integral of the first kind.

The simplest nontrivial flux is $q = 2$, i.e., $\gamma = \pi$, that is, in the language of random walks, the Stembridge case $\mathtt{Q} = -1$. It serves to demonstrate a common phenomenon, the appearance of various different-looking expressions for the same quantity related through combinatorial identities. For the generating function we have

$$\begin{aligned}
\mathcal{Z}(-1,t) &= \mathrm{Tr}\ \frac{1}{1 - t^2 H_\pi^2} = \mathrm{Tr}\ \frac{1}{1 - t^2(4 + C_2)} \\
&= \frac{1}{1-4t^2} \mathrm{Tr}\ \frac{1}{1 - \frac{t^2}{1-4t^2} C_2} = \frac{2}{\pi} \frac{1}{1-4t^2} K\left(\frac{4t^2}{1-4t^2} \right) \ ,
\end{aligned} \tag{29}$$

where we have used the fact that C_q is essentially the free Hamiltonian $H_{2\pi}$. Equivalently,

$$\begin{aligned}
\mathrm{Tr}\ H_\pi^{2n} &= \mathrm{Tr}\ (4 + C_2)^n = \sum_{k=0}^{[n/2]} \binom{n}{2k} 4^{n-2k} \mathrm{Tr}\ C_2^{2k} \\
&= \sum_{k=0}^{[n/2]} 2^{2n-4k} \binom{n}{2k} \binom{2k}{k}^2 .
\end{aligned} \tag{30}$$

An alternative formula is obtained by writing H_π^2 as a sum of two commuting parts,

$$H_\pi^2 = 4 + C_2 = 4 + x^2 + x^{-2} + y^2 + y^{-2} = (x + x^{-1})^2 + (y + y^{-1})^2 \tag{31}$$

and thus

$$\begin{aligned} \text{Tr } H_\pi^{2n} &= \text{Tr } \left[(x + x^{-1})^2 + (y + y^{-1})^2\right]^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \text{Tr } (x + x^{-1})^{2k} \, \text{Tr } (y + y^{-1})^{2(n-k)} \\ &= \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} , \end{aligned} \tag{32}$$

recovering Eq. (18).

Yet another formula is obtained by noticing that the value of the Casimir C_2 is between -4 and 4 (since x^2 and y^2 are phases) and thus it makes sense to put

$$\cos\phi = \frac{C_2}{4} \tag{33}$$

with some angle ϕ, it terms of which

$$H_\pi^2 = 4(1 + \cos\phi) = 8\cos^2\frac{\phi}{2} = 2\left(e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}}\right)^2 . \tag{34}$$

In this representation

$$H_\pi^{2n} = 2^n \left(e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}}\right)^{2n} = 2^n \sum_m \binom{2n}{m} e^{i(n-m)\phi} = 2^n \sum_m \binom{2n}{m} \cos(n-m)\phi , \tag{35}$$

where in the last step we used the fact that only the real part of $e^{i(n-m)\phi}$ contributes. Changing the variable $m = n + k$, $-n \le k \le n$, using the formula

$$\cos k\phi = \sum_{l=0}^{[\frac{k}{2}]} 2^{k-2l-1}(-1)^l \frac{k}{k-l} \binom{k-l}{l} (\cos\phi)^{k-2l} , \quad k \ge 1 \tag{36}$$

and using Eq. (33), we obtain

$$H_\pi^{2n} = \sum_{k=1}^{n} \sum_{l=0}^{[k/2]} 2^{n-k+2l}(-1)^l \frac{k}{k-l} \binom{2n}{n+k} \binom{k-l}{l} C_2^{k-2l} \tag{37}$$

(in the above, $\frac{k}{k-l}$ is defined to be 1 when both numerator and denominator are zero, as well as the ratios $\frac{2k}{2k-l}$ or $\frac{l}{l-s}$ below.) Taking the trace and using Eq. (24), only even k values will contribute, and we obain a third formula for $q = 2$:

$$\text{Tr } H_\pi^{2n} = \sum_{k=0}^{[n/2]} \sum_{l=0}^{k} (-1)^l \, 2^{n-2k+2l} \binom{2n}{n+2k} \frac{2k}{2k-l} \binom{2k-l}{l} \binom{2k-2l}{k-l}^2 . \tag{38}$$

It is remarkable that the above three different-looking formulas all agree and give the same result. The last formula may look a bit unmotivated and of dubious value, as it is a double sum (rather than a simple sum like the first two), but it is given in order to make contact with the upcoming results for higher q.

For the case $q = 3$, the derivation proceeds by solving the characteristic equation for $H_{2\pi/3}$, $\lambda^3 = 6\lambda + C_3$. The solutions are

$$\lambda_p = 2\sqrt{2}\cos\frac{\phi + 2p\pi}{3} \ , \quad p = 0, 1, 2 \tag{39}$$

with the angle ϕ now satisfying

$$\cos\phi = \frac{C_3}{4\sqrt{2}} \ . \tag{40}$$

Evaluating $\mathrm{Tr}\ H^n_{2\pi/3}$ amounts to averaging λ_p^n over the three values of p. In a calculation analogous to the third formula for $q = 2$ we have

$$\begin{aligned}\mathrm{Tr}\ H^{2n}_{2\pi/3} &= \frac{1}{3}\mathrm{Tr} \sum_{p=0,1,2} \lambda_p^{2n} = \frac{2^n}{3} \sum_{p=0,1,2} \mathrm{Tr}\ \left(\mathrm{e}^{\mathrm{i}\frac{\phi+2p\pi}{3}} + \mathrm{e}^{-\mathrm{i}\frac{\phi+2p\pi}{3}}\right)^{2n} \\ &= \frac{2^n}{3}\sum_{p,m}\binom{2n}{m}\mathrm{Tr}\ \mathrm{e}^{\mathrm{i}2(n-m)\frac{\phi+2p\pi}{3}} \ .\end{aligned} \tag{41}$$

The sum over p above will give zero unless $n - m$ is a multiple of 3. Putting $n - m = 3k$, we have

$$\mathrm{Tr}\ H^{2n}_{2\pi/3} = 2^n \sum_k \binom{2n}{n+3k}\mathrm{Tr}\ \mathrm{e}^{\mathrm{i}2k\phi} = 2^n \sum_k \binom{2n}{n+3k}\mathrm{Tr}\ \cos(2k\phi) \ . \tag{42}$$

The rest of the calculation proceeds in a way similar to the derivation of the last formula for $q = 2$. The result is

$$\mathrm{Tr}\ H^{2n}_{2\pi/3} = \sum_{k=0}^{[n/3]}\sum_{l=0}^{k}(-1)^l\, 2^{n-3k+3l}\binom{2n}{n+3k}\frac{2k}{2k-l}\binom{2k-l}{l}\binom{2k-2l}{k-l}^2 \ . \tag{43}$$

The similarity with (38) for $q = 2$ is obvious.

For $q = 4$ we proceed in a similar way. The solutions to the characteristic equation $\lambda^4 = 8\lambda^2 + 4 - C_4$ are

$$\begin{aligned}\lambda_p &= \pm\sqrt{4 \pm \sqrt{12 + C_4}} \\ &= 2\sqrt{2}\cos\frac{\phi + 2p\pi}{4} \ , \quad p = 0, 1, 2, 3\end{aligned} \tag{44}$$

with

$$\cos\phi = \frac{4 + C_4}{8} \ . \tag{45}$$

Expressing $\text{Tr } H^{2n}_{\pi/2} = \frac{1}{4}\sum_p \lambda_p^{2n}$ and choosing either the first or the second expression above for λ_p leads to two different-looking formulas. The first is

$$\text{Tr } H^{2n}_{\pi/2} = \sum_{k=0}^{[n/2]} \sum_{l=0}^{[k/2]} 2^{2n-2k-4l}\, 3^{k-2l} \binom{n}{2k}\binom{k}{2l}\binom{2l}{l}^2 . \tag{46}$$

The second one proceeds, again, in a way similar to the last of $q = 2$ and the $q = 3$ cases and gives

$$\text{Tr } H^{2n}_{\pi/2} = \sum_{k=0}^{[n/2]} \sum_{l=0}^{[k/2]} \sum_{s=0}^{[k/2]-l} (-1)^l\, 2^{n-4s} \binom{2n}{n+2k} \frac{k}{k-l} \binom{k-l}{l}\binom{k-2l}{2s}\binom{2s}{s}^2 . \tag{47}$$

This is similar to (38) and (43) but with a different structure. Clearly, the fact that $q = 4$ is not prime is related to the different appearance of the result.

The case $q = 5$ presents some qualitatively new features. It is the first case for which the characteristic equation involves non-rational coefficients. Further, since this equation is quintic, it has (in principle) no analytical solutions. We will present no explicit formula for $\text{Tr } H^{2n}_{2\pi/5}$ and leave its full treatment for a future publication.

Finally, we deal with the simpler $q = 6$ case. The solutions to the characteristic equation are now

$$\lambda_p = \pm 2\sqrt{1 + \sqrt{2}\cos\frac{\phi + 2p\pi}{3}}\ , \quad p = 0, 1, 2 \tag{48}$$

with

$$\cos\phi = \frac{36 + C_6}{32\sqrt{2}} . \tag{49}$$

A calculation along the lines of the previous ones gives

$$\text{Tr } H^{2n}_{\pi/3} = \sum_{k=0}^{[n/2]} \sum_{l=1}^{\left[\frac{n-2k}{3}\right]} \sum_{s=0}^{[l/2]} \sum_{t=0}^{[l/2]-s} (-1)^s\, 2^{2n-k-4l+5s-4t}\, 3^{2l-4s-4t} \binom{n}{2k+3l}\binom{2k+3l}{k} \frac{l}{l-s}\binom{l-s}{s}\binom{l-2s}{2t}\binom{2t}{t}^2 . \tag{50}$$

which is even more elaborate than the previous formulas.

This kind of computation can in principle be extended to higher values of q. Overall, we see no discernible pattern in the formulas obtained so far for $\text{Tr } H^{2n}_{2\pi/q}$. However, the fact that multiple not manifestly equivalent expressions exist for each q leaves the hope that a form more amenable to generalization may still exist.

5 Conclusion

To conclude, we have further explored the previously-considered generating function $Z_{m_1,m_2,l_1,l_2}(\mathrm{Q})$ of the probability distribution of the algebraic area of random walks on

a square lattice. We have: (i) elucidated the relation of $Z_{m,0,l,0}(\mathrm{e}^{2\mathrm{i}\pi/q})$ ("staircase walks") to the cyclic sieving phenomenon (by demonstrating that its values correspond to the numbers of subsets fixed by powers of the cyclic permutation); (ii) shown that the values of $Z_{m_1,0,l_1,l_2}(\mathrm{e}^{2\mathrm{i}\pi/q})$ ("biased walks") are integers under certain conditions, and explained why; (iii) obtained an explicit expression for $Z_{m_1,m_2,l_1,l_2}(-1)$ (Stembridge case), and connected it with the expression for the n-th moment of the Hofstadter Hamiltonian at the simplest nontrivial flux, $q = 2$; (iv) found explicit expressions for the same n-th moment for $q = 3, 4, 6$. Obtaining either a general closed expression for $Z_{m_1,m_2,l_1,l_2}(\mathrm{e}^{2\mathrm{i}\pi/q})$ or at least more particular cases thereof, as well as a general form of the n-th moment evaluated at any root of unity, are open tasks. Note that the values of $Z_{m_1,m_2,l_1,l_2}(\mathrm{e}^{2\mathrm{i}\pi/q})$ are seen numerically to be not integer anymore when $q \geq 5$ (to the exception of $q = 6$), making the task even more challenging.

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